

## Case Study – Launch an object in non-orbital flight

There are instances when has to analyze the behavior of an object that is launched from the ground. This could be a kid throwing a stone or a military unit firing an artillery gun or a hunter shooting in a forest. The typical trajectory of the projectile is given by Figure 1.

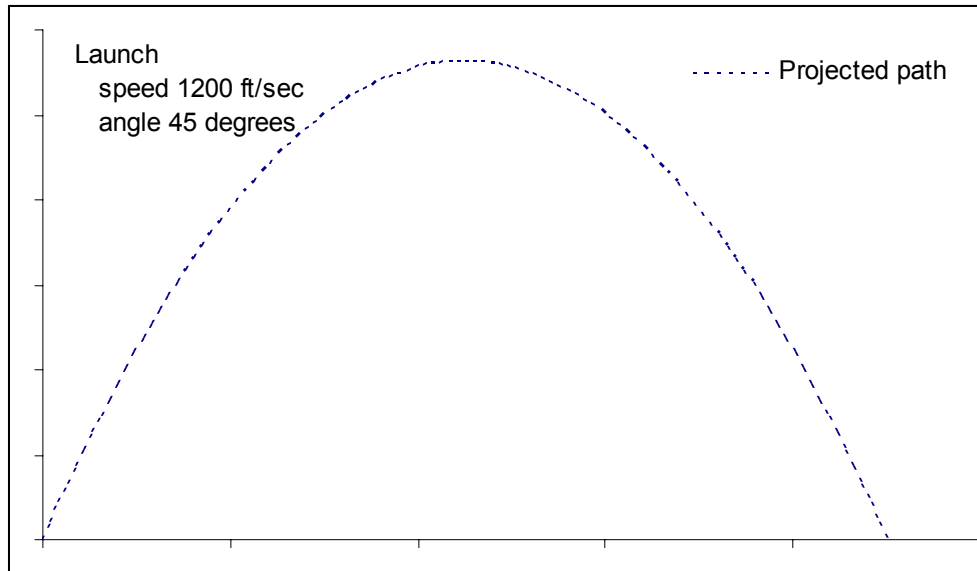


Figure 1

### Theory

The basic equations go way back – back to the time of Isaac Newton. The speed of an object,  $v$ , at time  $t$  is given by

$$v = v_0 + at \quad (1)$$

where  $v_0$  is the starting speed and  $a$  is the acceleration.

The distance an object travels,  $S$ , by time  $t$  is given by

$$S = v_0t + \frac{1}{2}at^2 \quad (2)$$

Next, consider some object that is launched with an initial speed of  $v_0$  in a direction given by the angle  $\theta$  with the horizontal (see Figure 2). The velocity can be decomposed into a horizontal component,  $v_{h0}$ , and a vertical component,  $v_{v0}$ , given by

$$v_{h0} = v_0 \cos(\theta) \quad \text{and} \quad v_{v0} = v_0 \sin(\theta)$$

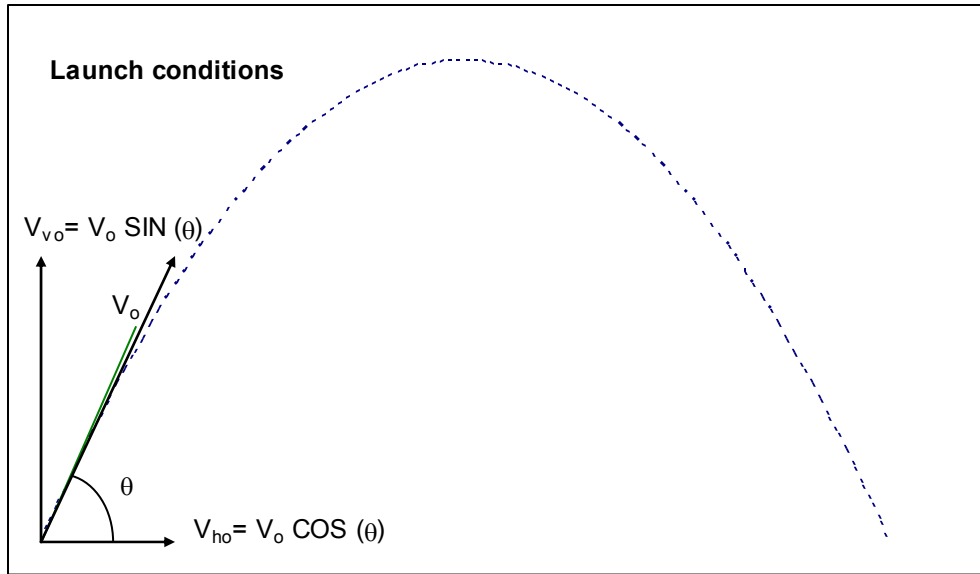


Figure 2

The vertical velocity determines how long the object will be in motion. It will go up against the pull of gravity until its vertical speed becomes zero. Then, it will spend exactly the same time coming down. Since the vertical speed at the top of the arc is zero, we can calculate the time it takes to reach the top. Let this be  $t_{top}$  and the acceleration due to gravity be  $g$ . Then, from (1), which is  $v = v_0 + at$ , we get  $0 = v_{v0} - gt_{top}$  or

$$t_{top} = \frac{v_{v0}}{g} \quad (3)$$

Given that the projectile takes  $t_{top}$  to get to the top of the arc, it will take the same amount of time coming down. Hence, it will be airborne for a total duration of  $2t_{top}$ . The vertical distance to the top is the distance it will travel in time  $t_{top}$ . Since the distance is given by (2), i.e.,

$$S = v_0 t + \frac{1}{2} a t^2 \text{ we get } S_{top} = v_{v0} t_{top} - \frac{1}{2} g t_{top}^2 = \frac{v_{v0}^2}{g} - \frac{1}{2} \frac{v_{v0}^2}{g} \text{ or}$$

$$S_{top} = \frac{1}{2} \frac{v_{v0}^2}{g} \quad (4)$$

In the horizontal direction, the projectile encounters no change in speed from its starting speed of  $v_{h0}$  (we assume that drag from air resistance is negligible). Hence, it will travel a total horizontal distance given by  $S_{far} = 2v_{h0}t_{top}$  or

$$S_{far} = 2v_{h0} \frac{v_{v0}}{g} \quad - (5)$$

Finally, we calculate the projectile's position at any time during its flight. The horizontal distance is easy to calculate since the acceleration is zero and the projectile starts with a speed of  $v_{h0}$ . Starting with (2), i.e.,  $S = v_0 t + \frac{1}{2} a t^2$ , we get

$$S_h = v_{h0} t \quad - (6)$$

The vertical position,  $S_v$ , is a little more difficult. During the first part of the flight, while the projectile is going up, the distance at time  $t$  is given by  $S_v = v_{v0} t - \frac{1}{2} g t^2$ . While it is coming down, the vertical distance is given by  $S_v = S_{top} - \frac{1}{2} g (t - t_{top})^2$ . Finally remember the projectile is going up from time zero to time  $t_{top}$ . So,

$$S_v = \begin{cases} v_{v0} t - \frac{1}{2} g t^2 & 0 \leq t \leq t_{top} \\ S_{top} - \frac{1}{2} g (t - t_{top})^2 & t_{top} \leq t \leq 2t_{top} \end{cases} \quad - (7)$$

### The calculations in Excel

We can directly map the equations listed above into Excel cells. The acceleration due to gravity is in B2. The launch speed and the angle of launch are in B3 and B4 respectively.

	A	B	C
1			
2	g		32 ft/sec <sup>2</sup>
3	V <sub>0</sub>		1200 ft/sec
4	θ		45°

Name these cells and others as we use them according to Table 1.

g	=Sheet1!\$B\$2
V0	=Sheet1!\$B\$3
theta	=Sheet1!\$B\$4
Vv0	=Sheet1!\$B\$5
Vh0	=Sheet1!\$B\$6
Ttop	=Sheet1!\$B\$7

Stop      =Sheet1!\$B\$8  
 Sfar      =Sheet1!\$B\$9  
 t          =Sheet1!\$A60

**Table 1**

Note that  $t$  uses a combined absolute-relative address whereas all the other names use only absolute references.

Continue with the various formulas developed above as in Figure 3. Note that the table showing the coordinates  $S_h$  and  $S_v$  of the object at time  $t$  starts in row 11 and continues to row 112. The formulas in the various cells are shown in Figure 4.

	A	B	C
5	$V_{vo}$	848.5281	ft/sec
6	$V_{ho}$	848.5281	ft/sec
7	$t_{top}$	26.5165	sec
8	$S_{top}$	11250	ft
9	$S_{far}$	45000	ft
10			
11	$t$	$S_h$	$S_v$
12	0	0	0
13	0.53033	450	445.5

**Figure 3**

	A	B	C
1			
2	g	32	ft/sec <sup>2</sup>
3	V <sub>o</sub>	1200	ft/sec
4	θ	45	°
5	V <sub>vo</sub>	=V0*SIN(RADIANS(theta))	ft/sec
6	V <sub>ho</sub>	=ROUND(V0*COS(RADIANS(theta)),12)	ft/sec
7	t <sub>top</sub>	=Vv0/g	sec
8	S <sub>top</sub>	=1/2*Vv0^2/g	ft
9	S <sub>far</sub>	=Vh0*2*Ttop	ft
10			
11	t	S <sub>h</sub>	S <sub>v</sub>
12	0	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
13	=A12+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
14	=A13+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
15	=A14+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
16	=A15+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
17	=A16+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
18	=A17+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
19	=A18+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)
20	=A19+2*Ttop/100	=Vh0*t	=IF(t<Ttop,Vv0*t-1/2*g*t^2,Stop-1/2*g*(t-Ttop)^2)

Figure 4

As the reader will notice, because we named the various cells, any subsequent references to those cells in other formulas is by name. Consequently, the formulas in the Theory section can be entered into the Excel worksheet using the same notation. It makes the worksheet very easy to understand, audit, and maintain. One formula probably bears explanation. The V<sub>ho</sub> formula includes a ROUND() function. This was necessary because, for some reason, Excel returns a small error in the COS function when the argument is Pi/2. Instead of returning zero, it returns a number of magnitude E-14.

#### Showing the result graphically

To show the trajectory as in Figure 1, we need to calculate the position of the projectile from time t=0 to time t=2\*t<sub>top</sub>. Then, the S<sub>h</sub> coordinates will be the x-values on a chart and the S<sub>v</sub> coordinates will be the y-values. That is done using the table in A11:C112. Column A slices the total flight duration (2\*t<sub>top</sub>) into 100 equi-spaced intervals. Columns B and C contain the corresponding S<sub>h</sub> and S<sub>v</sub> coordinates respectively. Then, B12:C112 are plotted in a XY Scatter chart to get the result in Figure 1.

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